The collection of all possible partitions of $[a,b]$ will be denoted by $\mathcal{P}[a,b]$ $\quad$ $\mathcal{P}$ = script P

Bounded Variation

Let f be defined on $[a,b]$. If $P = \{x_0, x_1, x_2, \ldots x_n\}$ is a partion of $[a,b]$ write,

$$\Delta f_k = f(x_k) - f(x_{k-1}), \text{ for } k = 1, 2, \ldots n$$

If there exists a positive number M,

Such that,

$$\sum_{k=1}^{n} |\Delta f_k| \le M \quad \text{for all partitions of } [a,b]$$

the f is said to be of bounded variation on $[a,b]$

[or]

We say that f is of bounded variation on $[a,b]$ if the set $\left\{ \sum_{k=1}^{n} |\Delta f_k| : P \in \mathcal{P}[a,b] \right\}$ is bounded above and l.u.b of this set is called total variation of f on $[a,b]$

Theorem :

If f is monotonic on $[a,b]$ then f is of bounded variation on $[a,b]$.

Proof:

It is enough to prove that ~~the~~ theorem when f is increasing. since f is decreasing then $-f$ is increasing

Assume that f increasing on $[a,b]$

Let $P = \{x_0, x_1, x_2, \ldots\ldots x_n\}$ be a partition of $[a,b]$

Then,

$\Delta f_k = f(x_k) - f(x_{k-1})$, $k = 1, 2, \ldots, n$

To prove that:

f is of bounded variation on $[a, b]$.

We have to prove that $\exists$ a positive number M, such that

$$\sum_{k=1}^{n} |\Delta f_k| \le M, \quad \forall \text{ partition } P \text{ of } [a, b]$$

Since f is increasing.

$$x_{k-1} < x_k$$

$$\Rightarrow f(x_{k-1}) \le f(x_k), \quad k = 1, 2, \ldots, n$$

$$\Rightarrow f(x_k) - f(x_{k-1}) \ge 0$$

$$\Rightarrow \Delta f_k \ge 0$$

$$\therefore \sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} \Delta f_k \qquad \{\because \Delta f_k \ge 0, \ |\Delta f_k| = \Delta f_k\}$$

$$= \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})]$$

$$= f(x_n) - f(x_0)$$

$$= f(b) - f(a)$$

$$\Rightarrow \sum_{k=1}^{n} |\Delta f_k| = f(b) - f(a)$$

This is true for every partition P of $[a, b]$

$\therefore f$ is of bounded variation on $[a, b]$.

Theorem:

If f is continuous on $[a, b]$ and if f' exists and is bounded in the interior, say $|f'|$ exists such that $|f'(x)| \le A$, $\forall x \in (a, b)$ then f is of bounded variation in $[a, b]$

Proof

Given that

i) f is continuous

ii) f' exists in (a,b)

iii) $|f'(x)| \le A$, $\forall x \in (a,b)$

W.P.T: f is bounded variation on $[a,b]$

Let $P = \{x_0, x_1, \ldots x_n\}$ be a partition of $[a,b]$

$$\Delta f_k = f(x_k) - f(x_{k-1})$$

From i) f is continuous on $[x_{k-1}, x_k]$

ii) f' is exists (x_{k-1}, x_k)

$\therefore$ By Lagrange mean value theorem

we have $\exists$ $\overset{c}{t_k} \in (\overset{a}{x_{k-1}}, \overset{b}{x_k})$. Such that

$$f'(t_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \qquad f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f'(t_k) = \frac{\Delta f_k}{\Delta x_k}$$

$$\Delta f_k = f'(t_k)\, \Delta x_k$$

$$|\Delta f_k| = |f'(t_k)\, \Delta x_k|$$

$$\sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} |f'(t_k)\, \Delta x_k|$$

$$\le \sum_{k=1}^{n} |f'(t_k)|\, |\Delta x_k|$$

$$\le \sum_{k=1}^{n} A\, |\Delta x_k|$$

$$= A \sum_{k=1}^{n} |\Delta x_k|$$

$$= A \sum_{k=1}^{n} (x_k - x_{k-1}) = A(x_n - x_0) = A(b-a)$$

$$\sum_{k=1}^{n} |\Delta f_k| \le A(b-a)$$

This is true for every partition $[a,b]$

f is bdd variation on $[a,b]$

Theorem [illegible]

If f is bounded variation on [illegible] say $\sum_{k=1}^{n} |\Delta f_k| \le M$ for all partition of $[a,b]$ then f is bounded on $[a,b]$

In fact $|f(x)| \le |f(a)| + M$ $\forall x \in [a,b]$

proof: Given f is of bounded variation on $[a,b]$

Say $\sum_{k=1}^{n} |\Delta f_k| \le M$ $\forall$ partition of $[a,b]$

P.T: 'f' is bounded on $[a,b]$

assume $x \in (a,b)$

Consider a partiation

$P = \{a, x, b\}$

$\because \sum_{k=1}^{n} |\Delta f_k| \le M$. for every partition of $[a,b]$

for this P, we have

$|f(x) - f(a)| + |f(b) - f(x)| \le M$

$\because$ each terms L.H.S is non negative

It is necessary that

$|f(x) - f(a)| \le M$

Also, $|f(x)| - f(a)| \le |f(x) - f(a)| \le M$

$|f(x)| - |f(a)| \le M$.

$\Rightarrow |f(x)| \le |f(a)| + M$, $\forall x \in (a,b)$

If $(x = a)$, $P = \{a, b\}$

If $x = b$, $P = \{a, b\}$

$|f(b) - f(a)| \le M$

$\Rightarrow |f(b)| \le |f(a) + M$ $\forall x \in [a,b]$

$\therefore |f(x)| \le f(a) + M$

$\therefore f$ is bounded variation on $[a,b]$

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Total Variation:

Let f be of bounded variation on $[a,b]$ and let $\Sigma(P)$ denotes the sum $\sum_{k=1}^{n} |\Delta f_k|$ corresponding to the partition

$$P = \{x_0, x_1, \dots x_n\} \text{ of } [a,b]$$

The number,

$$V_f(a,b) = \text{Sup}\{\Sigma(P) : P \in \mathcal{P}[a,b]\}$$ is called the total variation of 'f' on the Interval $[a,b]$.

Note:

* we write V_f Instead of $V_f(a,b)$
* f is of bounded variation on $[a,b]$, each sum $\sum_{k=1}^{n} |\Delta f_k| \geq 0$

$\therefore$ V_f is finite

also $V_f \geq 0$

* $V_f(a,b) = 0 \Leftrightarrow$ 'f' is constant on $[a,b]$

Theorem:

Assume that f and g are each of bounded variation on $[a,b]$ then so are their sum, difference and product are also bounded variation on $[a,b]$. are also we are

(i) $V_{f \pm g} \leq V_f + V_g$ and

ii) $V_{fg} \leq A V_f + B V_g$ where

(iii) $A = \text{Sup}\{|g(x)| : x \in [a,b]\}$ and

(iv) $B = \text{Sup}\{|f(x)| : x \in [a,b]\}$

Proof:

Given that, f and g are each of bounded variation on $[a,b]$

(i) To prove that: $f \pm g$ is of bounded variation on $[a,b]$ and $V_{f \pm g} \leq V_f + V_g$

Let $P = \{x_0, x_1, \dots x_n\}$ be an arbitrary partition of $[a,b]$

Let $h = f + g$

Then $\sum_{k=1}^{n} |\Delta h_k| = \sum_{k=1}^{n} |\Delta(f+g)_k|$

$$= \sum_{k=1}^{n} |(f+g)(x_k) - (f+g)(x_{k-1})|$$

$$= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1}) + g(x_k) - g(x_{k-1})|$$

$$\leq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})|$$

$$= \sum_{k=1}^{n} |\Delta f_k| + \sum_{k=1}^{n} |\Delta g_k|$$

$$\Rightarrow \sum_{k=1}^{n} |\Delta h_k| \leq \sum_{k=1}^{n} |\Delta f_k| + \sum_{k=1}^{n} |\Delta g_k|$$

$$\leq \sup\left\{\sum_{k=1}^{n} |\Delta f_k|\right\} + \sup\left\{\sum_{k=1}^{n} |\Delta g_k|\right\}$$

$$\Rightarrow \sum_{k=1}^{n} |\Delta h_k| \leq V_f + V_g$$

$\therefore \left\{\sum_{k=1}^{n} |\Delta h_k|\right\}$ is bounded above

$\therefore$ h is of bounded variation on $[a,b]$

i.e) $f+g$ is of bounded variation on $[a,b]$

Also, $\sup\left\{\sum_{k=1}^{n} |\Delta h_k|\right\}$ exists

i.e) $V_h \leq V_f + V_g$ exists and it is true for all $P \in \mathcal{P}[a,b]$

i.e) V_{f+g} is an supremum and

$V_f + V_g$ is an upper bound

$\therefore V_{f+g} \leq V_f + V_g$

Similarly, $f-g$ is of bounded variation on $[a,b]$ and

$$V_{f-g} \leq V_f + V_g$$

To prove that $f \cdot g$ is of bounded variation on $[a,b]$

and $V_{fg} \leq AV_f + BV_g$

Let $h = f \cdot g$

Then for every partition P of $[a,b]$ we have,

$$|\Delta h_k| = |h(x_k) - h(x_{k-1})|$$

$$= |(f \cdot g)(x_k) - (f \cdot g)(x_{k-1})|$$

$$\Rightarrow |\Delta h_k| = |f(x_k) g(x_k) - f(x_{k-1}) \cdot g(x_{k-1})|$$

+ & - by $f(x_k) g(x_{k-1})$

$$= |f(x_k) \cdot g(x_k) - f(x_k) g(x_{k-1}) + f(x_k) g(x_{k-1}) - f(x_{k-1}) \cdot g(x_{k-1})|$$

$$= |f(x_k) \{ g(x_k) - g(x_{k-1}) \} + g(x_{k-1}) \{ \cancel{f(x_{k-1}) - g(x_{k-1})} \}|$$

$$\leq |f(x_k)| \, |g(x_k) - g(x_{k-1})| + |g(x_{k-1})| \cdot \cancel{+ |f(x_{k-1})| \cdot |g(x_{k-1})|}$$

$$\Rightarrow |\Delta h_k| = B |\Delta g_k| + A |\Delta f_k|$$

$$\Rightarrow \sum_{k=1}^{n} |\Delta h_k| \leq B \sum_{k=1}^{n} |\Delta g_k| + A \sum_{k=1}^{n} |\Delta f_k|$$

$$\leq B \sup \left\{ \sum_{k=1}^{n} |\Delta g_k| : P \in \mathcal{P}[a,b] \right\} + A \sup \left\{ \sum_{k=1}^{n} |\Delta f_k| : P \in \mathcal{P}[a,b] \right\}$$

$$= BV_g + AV_f$$

$$\Rightarrow \sum_{k=1}^{n} |\Delta h_k| \leq BV_g + AV_f \geq 0, \ \forall \ P \in \mathcal{P}[a,b]$$

$\Rightarrow \left\{ \sum_{k=1}^{n} |\Delta h_k| : P \in \mathcal{P}[a,b] \right\}$ is bounded above by

$AV_f + BV_g$

$\therefore$ h is of bounded variation

i.e.) fg is of bounded variation

$\therefore \sup \left\{ \sum_{k=1}^{n} |\Delta h_k| : P \in \mathcal{P}[a,b] \right\}$ exists

ii) V_{fg} exists

Since l.u.b $\le$ u.b

$V_{fg} \le A V_f + B V_g$

Hence the proof

Quotients were not including foregoing theorem because the reciprocal of a function of bounded variation need not be of bounded variation.

For eg: If $f(x) \to 0$ as $x \to x_0$ then $\frac{1}{f}$ will not be bounded on any interval containing x_0

And $\frac{1}{f}$ cannot be bounded variation on such an interval

To extend the above theorem to quotient it suffices to exclude functions whose value become arbitrarily close to zero.

Theorem : 10

Let f be of bounded variation on $[a,b]$ and assume that f is bounded away from zero.

ie) Suppose that there exists a positive number m such that $0 < m \le |f(x)|$

then $g = \frac{1}{f}$ is also of bounded variation on $[a,b]$ and $V_g \le \frac{V_f}{m^2}$

proof:

Given that : (i) f is of bounded variation on $[a,b]$.

then $\sum_{k=1}^{n} |\Delta f_k| \le V_f$

(ii) f is bounded away from zero.

ie) $\exists$ $m>0$ such that $m \le |f(x)|$

To prove that: $g = \frac{1}{f}$ is of bounded variation on $[a,b]$

Let $P = \{x_0, x_1, x_2, \ldots x_n\}$ be an arbitrary partition of $[a,b]$.

Let $g = \frac{1}{f}$

Then, $|\Delta g_k| = |g(x_k) - g(x_{k-1})|$

$$= \left| \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})} \right|$$

$$= \left| \frac{f(x_{k-1}) - f(x_k)}{f(x_k)\, f(x_{k-1})} \right|$$

$$= \frac{|f(x_k) - f(x_{k-1})|}{|f(x_k)|\, |f(x_{k-1})|}$$

$$= \frac{|\Delta f_k|}{|f(x_k)|\, |f(x_{k-1})|}$$

$\left\{ \because m \le |f(x)|,\ \forall x \in [a,b] \right.$

$\left. \frac{1}{m} \ge \frac{1}{f(x)},\ \forall x \in [a,b] \right\}$

$$\Rightarrow |\Delta g_k| \le \frac{|\Delta f_k|}{m \cdot m}$$

$$\Rightarrow |\Delta g_k| \le \frac{|\Delta f_k|}{m^2}$$

$$\Rightarrow \sum_{k=1}^{n} |\Delta g_k| \le \frac{1}{m^2} \sum_{k=1}^{n} |\Delta f_k|$$

$$\le \frac{1}{m^2} \cdot V_f$$

$\because V_f = \sup \{ \sum_{k=1}^{n} |\Delta f_k|$

$\therefore \left\{ \sum_{k=1}^{n} |\Delta g_k| : P \in \mathcal{P}[a,b] \right\}$ is bounded above by $\frac{V_f}{m^2}$.

$\therefore g$ is of bounded variation on $[a,b]$

$\Rightarrow \frac{1}{f}$ is of bounded variation on $[a,b]$

$\therefore \quad \text{Sup}\left\{\sum_{k=1}^{n} |\Delta g_k| : P \in \mathcal{P}[a,b]\right\}$ exists

ie) $V_g = V_{1/f}$ exists

Since $l\cdot u\cdot b \le u\cdot b$ 17

$$\text{Sup}\left\{\sum_{k=1}^{n} |\Delta g_k| : P \in \mathcal{P}[a,b]\right\} \le \frac{V_f}{m^2}$$

$$\Rightarrow V_g \le \frac{V_f}{m^2}$$

Hence the proof.

Additive property of Total Variation:

Theorem: 12

Let f be a bounded variation on $[a,b]$ and assume that $c \in (a,b)$. then f is bounded variation on $[a,c]$ and on $[c,b]$ and we have $V_f(a,b) = V_f(a,c) + V_f(c,b)$

Proof:

Given that,

f is of bounded variation on $[a,b]$

To prove that: f is of bounded variation on $[a,c]$ and $[c,b]$

Let P_1 be a partition of $[a,c]$

Let P_2 be a partition of $[c,b]$

Then $P_0 = P_1 \cup P_2$ is a partition of $[a,b]$

Since f is of bounded variation on $[a,b]$

$$\sum_{k=1}^{n} |\Delta f_k| \le V_f(a,b)$$

$\left[\because V_f(a,b) = \text{Sup}\{\Sigma|\Delta f_k|, P \in \mathcal{P}[a,b]\} \Rightarrow \Sigma|\Delta f_k| \le V_f(a,b)\right]$

If $\Sigma(P)$ denotes the sum $\sum_{k=1}^{n} |\Delta f_k|$ corresponding to the partition P of appropriate interval.

We can write,

$$\Sigma(P_1) + \Sigma(P_2) = \Sigma(P_0)$$
$$\le \text{Sup}\{\Sigma(P_0) : P_0 \in \mathcal{P}[a,b]\}$$